1 Topics

Chapter 1: Unbounded Operators

1.1: Domains, Graphs, Adjoints, and Spectrum

1.2: Symmetric and Self-Adjoint Operators: The Basic Criterion for Self-Adjointness

1.3: The Spectral Theorem

1.4: Stone's Theorem

Chapter 2: Locally Convex Spaces

2.1: Fréchet Spaces

2.2: Schwartz Space and the Tempered Distributions

2.3: Inductive Limits: Generalized Functions and Weak Solutions to PDEs

2.4: Hölder and Sobolev Spaces

Chapter 3: The Fourier Transform

3.1: The Fourier Transform of $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ and Convolutions

3.2: The Range of the Fourier Transform: Classical Spaces

3.3: Solving PDEs with Transform Methods

Chapter 4: Introduction to C^*-algebras

4.1: Definitions and Examples

4.2: The Riesz Functional Calculus

4.3: Abelian C^*-algebras and the Functional Calculus

4.4: The Positive Elements in an C^*-algebra

4.5: Ideals in a C^*-algebra

4.6: Representations of a C^*-algebra

4.7: Positive Linear Functionals and the GNS Construction

Chapter 5: Special Classes of Operators

5.1: Trace Class and Hilbert-Schmidt Operators

5.2: Fredholm Operators and Behavior of the Spectrum Under Compact Pertubations

2 Homework Problems

1. Define

$$A = i\frac{d}{dx}$$

on $L^2[0,1]$. Define two domains for A: $\mathcal{D}_1(A) = AC[0,1]$ and $\mathcal{D}_2(A) = \{f \in AC[0,1] : f(0) = 0\}$ Prove that A is closed on both domains.

2. Let A and B be operators defined on a Hilbert space. Show that $(\alpha A)^* = \overline{\alpha} A^*$ for scalar α. Moreover show $A^* + B^* \subset (A+B)^*$ where $\mathcal{D}(A^* + B^*) = \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$. Show that $(A+B)^* = A^* + B^*$ only if one of the operators is bounded. Give an example where equality doesn't hold.

3. Let A and B be operators defined on a Hilbert space such that AB is densely defined. Prove that $(AB)^* \supset B^* A^*$. Moreover if B is bounded then show $(BA)^* = A^* B^*$.

4. An alternative way to define a normal operator(to allow for unboundedness) is the following: A is called **normal** if $\|Af\| = \|A^* f\|$ for all $f \in \mathcal{D}(A) = \mathcal{D}(A^*)$. Prove that if A is normal then so is $A + z$ for all $z \in \mathbb{C}$.

5. Using the above definition, prove that normal operators are always closed.

6. Define

$$A = -i\frac{d}{dx}$$

on $L^2[0, 2\pi]$ with domain $\mathcal{D}(A) = \{f \in C^1[0, 2\pi] : f(0) = f(2\pi) = 0\}$. Prove that A is symmetric. Find A^* and its respective domain $\mathcal{D}(A^*)$. Is A self-adjoint? If not find a self-adjoint extension of A.

7. (a) Suppose that B is a symmetric operator such that $A \subset B$ and that $\mathrm{Ran}(A+i) = \mathrm{Ran}(B+i)$. Prove that $B = A$.
(b) Suppose A is a symmetric operator with $\mathrm{Ran}(A+i) = \mathcal{H}$ but $\mathrm{Ran}(A-i) \neq \mathcal{H}$. Prove that A has no self-adjoint extensions.

8. Define

$$A = -\frac{d^2}{dx^2}$$

on $L^2(\mathbb{R})$ with domain $\mathcal{D}(A) = C_0^\infty(\mathbb{R})$. Compute A^* and its respective domain $\mathcal{D}(A^*)$. Is A essentially self-adjoint?

9. Let M_g be the multiplication operator by g. Find two dense linear subspaces of $L^2(\mathbb{R})$, D_1 and D_2 with $D_1 \cap D_2 = \{0\}$ so that M_x is essentially self-adjoint on D_1 and M_{x^2} is essentially self-adjoint on D_2.

10. Suppose A is self-adjoint and B is any operator such that $\|B - z_0\| \leq r$ for some $z_0 \in \mathbb{C}$ and $r > 0$. Show that $\sigma(A + B) \subset \sigma(A) + \overline{B_r(z_0)}$ where $B_r(z_0)$ is the ball centered at z_0 with radius r.

11. Let A be an self-adjoint operator and let P_A be its corresponding projection valued measures. Prove that:

$$\sigma(A) = \{\lambda \in \mathbb{R} : P_A(\lambda - \varepsilon, \lambda + \varepsilon) \neq 0, \text{ for all } \varepsilon > 0\}.$$

12. Let A be a closed operator and set $|A| = \sqrt{A^*A}$. Prove that $\||A|f\| = \|Af\|$. Deduce that $\mathrm{Ker}(A) = \mathrm{Ker}(|A|) = \mathrm{Ran}(|A|)^\perp$ and that

$$U = \begin{cases} g = |A|f \mapsto Af & \text{if } g \in \mathrm{Ran}(|A|) \\ g \mapsto 0 & \text{if } g \in \mathrm{Ker}(|A|) \end{cases}$$

extends to a well-defined partial isometry. Conclude that $A = U|A|$. (This is an extension of the polar decomposition for not necessarily bounded operators)

13. Let A be a self-adjoint operator. Prove that the resolvent operator can be realized as the following representation:

$$R_A(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, dP_\lambda\,.$$

Deduce that the quadratic form $F_\psi(z) = \langle \psi, R_A(z)\psi\rangle$ is a holomorphic function from the upper half plane to itself. (These types of functions are called **Herglotz** functions and this process is called the **Borel** transform of measures.)

14. Consider the one-parameter unitary group given by $U(t)f(x) = f(x - t \bmod 2\pi)$ for $f \in L^2(0, 2\pi)$. What is the generator of U?

15. Suppose M is the Riemann surface of $\sqrt{z}$ and $\mathcal{H} = L^2(M)$ with Lebesgue measure locally. Define the following two operators:

$$A = -i\frac{\partial}{\partial x} \text{ and } B = -i\frac{\partial}{\partial y}$$

both with domain $\mathcal{D} = \{f \in C^\infty(M) : \text{supp}(f) \text{ is compact },0 \notin \text{supp}(f)\}$. Prove that both A and B are essentially self-adjoint, both $A, B : \mathcal{D} \to \mathcal{D}$, and $(AB)f = (BA)f$ for all $f \in \mathcal{D}$. However show that $\exp(it\overline{A})$ and $\exp(it\overline{B})$ do not commute.

16. Let μ be a finite complex Borel measure on $\mathbb{R}$ and let

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{-it\lambda}\, d\mu(\lambda)\,.$$

Prove that

$$\lim_{c\to\infty} \frac{1}{c} \int_0^c |\hat{\mu}(t)|^2\, dt = \sum_{\lambda\in\mathbb{R}} |\mu(\{\lambda\})|^2$$

where the sum on the right hand side is finite.

17. Let A be a self-adjoint operator. Using the previous problem, prove that $\lim_{t\to\infty} U(t) = 0$ where $U(t) = \exp(-itA)$.

18. Let X be an infinite dimensional locally convex space and X^* its dual space. Prove that no weakly continuous semi-norm is actually a norm. (This shows semi-norms are essential tools.)

19. Let $\{\rho_\alpha\}_{\alpha\in A}$ be a family of semi-norms so that some finite sum $\rho_{\alpha_1}+\cdots+\rho_{\alpha_n}$ is actually a norm. Prove that $\{\rho_\alpha\}_{\alpha\in A}$ is equivalent to a family of norms.

20. (Delta "function") Let $b \in \mathbb{R}$. Define δ_b to be the linear functional $\delta_b(f) = f(b)$ for $f \in \mathscr{S}(\mathbb{R})$. Prove that $\delta_b \in \mathscr{S}'(\mathbb{R})$, but there is no function $g(x)$ for which

$$\delta_b(f) = \int_{\mathbb{R}} g(x)f(x)\,dx$$

for all $f \in \mathscr{S}(\mathbb{R})$. However prove that there is a measure μ_b such that

$$\delta_b(f) = \int_{\mathbb{R}} f(x)\,d\mu_b(x)\,.$$

21. Let δ_b be defined as above. Prove that $\delta_b'(f) = -f(b)$ in the distributional sense.

22. The Cauchy principle part integral is given by:

$$\mathscr{P}\left(\frac{1}{x}\right) : f \mapsto \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} \frac{1}{x} f(x)\, dx\,.$$

Prove that $\mathscr{P}(1/x)$ is a distribution and the following limit holds in the weak topology sense on $\mathscr{S}'$ to $\mathscr{S}$:

$$\lim_{\varepsilon \downarrow 0} \frac{1}{x - x_0 + i\varepsilon} = \mathscr{P}\left(\frac{1}{x - x_0}\right) - i\pi\delta(x - x_0)\,.$$

23. Consider the following initial value problem:

$$\begin{cases} u_t + \dfrac{1}{2}\|\nabla u\|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \|x\| & \text{on } \mathbb{R}^n \times \{t = 0\}\,. \end{cases}$$

Define the following function:

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ \frac{\|x - y\|^2}{2t} + \|y\| \right\}.$$

Prove that u is a weak solution to the PDE.

24. Prove that the Hölder space, $C^{k,\gamma}(\overline{\Omega})$, is a Banach space.

25. Suppose $u, v \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$.
(a) Prove that $D^\alpha \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi-indicies α, β with $|\alpha| + |\beta| \leq k$.
(b) Prove that for each $\lambda, \mu \in \mathbb{R}$, one has $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ with $|\alpha| \leq k$.
(c) Prove that if $U \subset \Omega$ open, then $u \in W^{k,p}(U)$.
(d) Prove that if $w \in C_c^\infty(\Omega)$ then $wu \in W^{k,p}$ and

$$D^\alpha(wu) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta w D^{\alpha-\beta} u\,.$$

26. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1|, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0,\ |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0,\ |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0,\ |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0,\ |x_1| < -x_2\,. \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{k,p}(\Omega)$.

27. Given $\Omega \subset \mathbb{R}^n$, prove the following interpolation inequality:

$$\int_\Omega \|Du\|^p \, dx \leq C \left(\int_\Omega |u|^p \, dx \right)^{1/2} \left(\int_\Omega \|D^2 u\|^p \, dx \right)^{1/2}$$

for $2 \leq p < \infty$ and all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ where C is some constant.

28. Compute the Fourier transform of $\mathscr{P}(1/x)$, the Cauchy principle part.

29. Let $f \in \mathscr{S}(\mathbb{R}^n$, R a rotation, and define a map $D_\lambda x = \lambda x$ on $\mathbb{R}^n$.

(a) Prove that $\widehat{f \circ R} = \hat{f} \circ R^t$, where R^t is the transpose of R.

(b) Prove that $\widehat{f \circ D_\lambda} = \lambda^{-n} \hat{f} \circ D_{\lambda^{-1}}$.

(c) Let $T \in \mathscr{S}'(\mathbb{R}^n)$ and prove that $\widehat{T \circ R} = \hat{T} \circ R^t$ and $\widehat{T \circ D_\lambda} = \lambda^{-n} \hat{T} \circ D_{\lambda^{-1}}$.

30. Find a function $f(x)$ that satisfies all the conditions in the definition of a function of positive type except continuity. To which function of positive type is $f(x)$ equal a.e.?

31. Solve the following initial value problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\}\,. \end{cases}$$

32. Given $d > 0$, solve the following initial value problem:

$$\begin{cases} u_{tt} + 2du_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R} \times \{t=0\}\,. \end{cases}$$

33. Let $\mathbb{D}$ be the closed unit disk in $\mathbb{C}$ and let $\mathcal{A} = \{f \in C(\mathbb{D}) : f \text{ is analytic on } \mathbb{D}\}$. For $f \in \mathcal{A}$, define $f^*(z) = \overline{f(\overline{z})}$. Show that $f \mapsto f^*$ is an involution on $\mathcal{A}$ and $\|f^*\| = \|f\|$ for all $f \in \mathcal{A}$, but $\mathcal{A}$ is not a C^*-algebra.

34. Let $\mathcal{A}$ be a C^*-algebra and let X be a locally compact space. Show that:

$$C_0(X,\mathcal{A}) = \{f \in C(X,\mathcal{A}) : \text{ for every } \varepsilon > 0 \ \{x : \|f(x)\| \geq \varepsilon\} \text{ is compact}\}$$

is a C^*-algebra, where all the operations are defined pointwise and the norm is the sup norm, that is $\|f\| = \sup\{\|f(x)\| : x \in X\}$

35. Using the notation from the previous problem, show that if X and Y are locally compact spaces, then there is a natural $*-$isomorphism from $C_0(X, C_0(Y))$ onto $C_0(X \times Y)$.

36. Let $\mathcal{A}$ be a Banach algebra with identity. Prove that if $a \in \mathcal{A}$ and $f \in \mathrm{Hol}(a)$, then $\sigma(f(a)) = f(\sigma(a))$.

37. Prove that if $\mathcal{A}$ is a Banach algebra with identity, $a \in \mathcal{A}$, $f \in \mathrm{Hol}(a)$, and g is analytic in a neighborhood of $f(\sigma(a))$, then $g \circ f \in \mathrm{Hol}(a)$ and $g(f(a)) = (g \circ f)(a)$.

38. Let X be a compact subset of $\mathbb{C}$. Prove that $C(X)$ is a C^*-algebra with a single generator.

39. A projection in a C^*-algebra is a Hermitian idempotent, that is $a^* = a$ and $a^2 = a$. If X is a compact space, show that X is totally disconnected if and only if $C(X)$, as a C^*-algebra, is generated by its projections.

40. If $\mathcal{A}$ is a unital C^*-algebra and $a \in \text{Re } \mathcal{A}$, show that $U = e^{ia}$ is a unitary. Is the converse true?(Be sure to justify.)

41. Let $\mathcal{A}$ be a C^*-algebra. We say $a \in \mathcal{A}$ has a **logarithm**, if there is a $b \in \mathcal{A}$ such that $e^b = a$. If $a \in \mathcal{A}_+$, show that a has a positive logarithm if and only if a is invertible. Moreover show that an invertible Hermitian element always has a logarithm. Is this logarithm Hermitian?

42. Let $\mathcal{A}$ be a C^*-algebra, show that if $a \in \text{Re } \mathcal{A}$, then $|a| = a_+ + a_-$.

43. Give an example of a C^*-algebra $\mathcal{A}$ and two positive elements, a, $b \in \mathcal{A}$ such that ab is not positive. If $ab = ba$, show that $ab \in \mathcal{A}_+$.

44. Let $\mathcal{A}$ be a C^*-algebra. Show that each element $a \in B_{\mathrm{Re}\mathcal{A}}$ is the sum of two unitaries in $\mathcal{A}$.

45. If $\mathcal{A}$ is a C^*-algebra, $\mathcal{I}$ is a closed ideal of $\mathcal{A}$, and $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, show that the C^*-algebra generated by $\mathcal{I} \cup \mathcal{B}$ is $\mathcal{I} + \mathcal{B}$.

46. If $\mathcal{A}$ is a C^*-algebra and $\mathcal{I}$ and $\mathcal{J}$ are closed ideals of $\mathcal{A}$, show that $\mathcal{I} + \mathcal{J}$ is a closed ideal of $\mathcal{A}$.

47. Let $\mathbb{D}$ be the closed unit disk in $\mathbb{C}$. Give an example of a non-closed ideal of $C(\mathbb{D})$ that is not self-adjoint.

48. Let (X, Ω, μ) be a $\sigma-$finite measure space and M_φ be the multiplication operator by φ in $L^2(\Omega)$. Recall that $\pi : L^\infty(\Omega) \to \mathcal{B}(L^2(\Omega))$ defined by $\pi(\varphi) = M_\varphi$ is a representation. Prove that π is cyclic and that a $f \in L^2(\Omega)$ is a cyclic vector for π if and only if f does not vanish on a set of positive μ measure.

49. Let $\mathcal{A}$ be any C^*-algebra contained in $\mathcal{B}(\mathcal{H})$ that contains the compact operators. If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is the identity representation, show that $\pi^{(\infty)}$ is a cyclic representation.

50. Let $\mathcal{A}$ be a C^*-algebra, $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation, and $\mathcal{I}$ an ideal of $\mathcal{A}$. Define $\mathcal{H}_{\mathcal{I}} = \overline{\text{span}\{\rho(\mathcal{I})\mathcal{H}\}}$ and let $\rho_{\mathcal{I}} : \mathcal{I} \to \mathcal{B}(\mathcal{H}_{\mathcal{I}})$ by $\rho_{\mathcal{I}}(x) = \rho(x)|_{\mathcal{H}_{\mathcal{I}}}$. Show that $\rho_{\mathcal{I}}$ is a non-degenerate representation of $\mathcal{I}$. What is its extension?

51. Show that $\varphi_0 : \mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H}) \to \mathbb{C}$ is a state if and only if there is a state $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ such that $\varphi_0(T + \mathcal{B}_0(\mathcal{H})) = \varphi(T)$ for every $T \in \mathcal{B}(\mathcal{H})$.

52. Show that an element a in a C^*-algebra is positive if and only if $\varphi(a) \geq 0$ for every state φ.

53. Let $\mathcal{A} = M_{nn}(\mathbb{C})$. Define $\tau : \mathcal{A} \to \mathbb{C}$ by $\tau(a) = \text{tr}(a)$. Prove that τ is a state on $\mathcal{A}$ and construct the Hilbert space via the GNS construction. (Don't forget to identify the representation.)

54. If $A \in \mathcal{B}(\mathcal{H})$, show that A is trace class if and only if

$$\sum_{\mathcal{E}} |\langle Ae, e\rangle| < \infty$$

for every orthonormal basis $\mathcal{E}$. Give an example of an orthonormal basis and operator A that is not trace class but such that this series converges absolutely for this particular basis.

55. Let $g,\, h \in \mathcal{H}$. Recall the finite rank operator $g \otimes h$ defined on $\mathcal{H}$ is given by $(g \otimes h)(f) = \langle f, g\rangle h$ for $f \in \mathcal{H}$. Show that $\|g \otimes h\|_{\text{op}} = \|g \otimes h\|_1 = \|g \otimes h\|_2 = \|g\|\|h\|$.

56. For $p > 0$, define the following class of operators: $\mathcal{B}_p(\mathcal{H}) = \{A : |A|^{p/2} \in \mathcal{B}_2(\mathcal{H})\}$. This class is called the **Schatten** $p-$**class**. If $A \in \mathcal{B}_p(\mathcal{H})$, let $\|A\|_p = \||A|^{p/2}\|^{2/p}$. Prove that $\mathcal{B}_p(\mathcal{H})$ is a vector space and a Banach space.

57. Let T be a closed Fredholm operator with compact resolvent. Prove that $\sigma_c(T) = \emptyset$. Is it necessary that the resolvent be compact? (Be sure to justify.)

58. Let T be a Fredholm operator with $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$ both non-zero. Show that for every $\varepsilon > 0$, there is a Fredholm operator S with $\|S - T\| < \varepsilon$, $\dim \mathrm{Ker}(S) < \dim \mathrm{Ker}(T)$, and $\dim \mathrm{Ker}(S^*) < \dim \mathrm{Ker}(T^*)$. Can these inequalities be reversed?

59. If $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$ is the natural map, show that if $A \in \mathcal{B}(\mathcal{H})$ and $\pi(A)$ is Hermitian, then there is a self-adjoint operator B such that $A - B \in \mathcal{B}_0(\mathcal{H})$. If $\pi(A)$ is positive, show there is a positive operator B such that $A - B \in \mathcal{B}_0(\mathcal{H})$.